THE CONIC SECTIONS

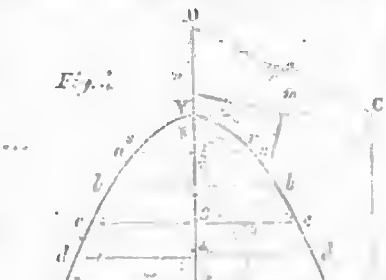
CONSIDERED IN REFERENCE TO THEIR PRACTICAL APPLICATIONS.

The Parabola.

At page 493 of the present volume of THE BUILDER, we gave a method of describing the curve of a parabola by means of points, which points were obtained from a series of ordinates computed by the rule to equation (A), or that to equation (C), both these rules being applicable to the same purpose. This method of construction is sufficiently satisfactory in practice, and practical men adopt it in preference to other methods, in consequence of its being easily understood, and more in accordance with their usual routine of operation; it is, however, attended with greater mental and manual labour than could be wished for, since the ordinates have first to be computed numerically, and afterwards successively applied to a scale of equal parts, before the positions of the several points through which the curve has to pass can be assigned, and if these processes be not performed with considerable accuracy, the resulting figure may be very far different from the true one. In order, therefore, to avoid the errors incident to this mode of construction, we shall effect the operation in another way, by which the computation of the ordinates is dispensed with, as well as the application of the computed results to a scale of equal parts, and thence to the several ordinates which are supposed to be previously drawn. By this method it is necessary that the parameter of the axis and the position of the focus shall both be known, or rather, the position of the focus, and that of a point in the axis produced, through which the directrix of the curve is made to pass; but since both these points depend on the parameter, the magnitude of that element must first of all be ascertained. Now, it has already been stated, page 463, that the parameter of the axis is a third proportional to any abscissa and its corresponding ordinate, and we may here add, that the distance between the focus and vertex of the curve, and also the distance between the vertex of the curve and that point in the axis produced, through which the directrix passes, is equal to one-fourth of the parameter, hence the method of determining the positions of these points is obvious, and is as follows:—

Let AB, fig. 4, be the base of the required parabola, and let AB be bisected perpendicularly in E by the straight line ED, and make EV equal to the axis of the curve. Upon the axis EV, and the semi-base EB, describe the rectangular parallelogram EBCV, and draw

Fig. 4.



the diagonal EC; at the point C, erect the perpendicular CD, meeting the production of the axis in the point D; then is VD the parameter of the curve to the axis, EV.

Bisect VC in m, and at the point m erect the perpendicular mn, meeting the production of the axis in the point n, and make VF equal to Vn; then is F the focus and n the point in the axis through which the directrix of the curve must pass.

Since the triangle ECD is rightangled at C by the construction, and CV perpendicular to ED, it follows from the eighth proposition of the sixth book of Euclid's "Elements of Geometry," that the triangles ECV and CDV are similar to one another, and consequently the homologous sides are proportional; thus we have

$$EV : VC :: VC : VD = \frac{VC^2}{EV},$$

so that VD is a third proportional to the abscissa VE, and the ordinate EB; but by the definitions the parameter of the curve to the axis VE, is a third proportional to any abscissa and its corresponding ordinate; consequently, VD is the parameter; thus far, therefore, the construction is accurate.

Again, since the triangle Emn is rightangled at m, and mV perpendicular to En, the triangles EmV and mnV are similar to one another, and their homologous sides are proportional; hence we have

$$EV : Vm :: Vm : Vn = \frac{\overline{Vm}^2}{EV};$$

from which it appears that Vn is a third proportional to EV and Vm; but Vm=½ VC by construction, consequently $\overline{Vm}^2 = \frac{1}{4}VC^2$, and by substitution it is $Vn = \frac{VC^2}{4EV}$; so that Vn and VF are each of them equal to one-fourth of the parameter VD.

Through the point F draw the straight line rs parallel to AB, and make Fr and Fs respectively equal to Fn, or one-half of VD; then r and s are points in the curve, and rs is the parameter to the axis VE.

Let the axis VE be divided into any number of equal parts in the points 1, 2, 3, 4, &c., the more numerous the points of division, the more correctly will the curve be delineated; and through the several points thus determined, and parallel to the base AB, draw the series of double ordinates aa, bb, cc, dd, &c.; then from the focus F, with the several distances n 1, n 2, n 3, n 4, &c., intersect the ordinates both ways in the points a, a; b, b; c, c; d, d, &c., and these will respectively be points in the curve; then with a fine pen and a steady hand let a line be drawn through all the points, and the line thus traced will be the curve of the parabola.

Having thus effected the delineation of the curve, it may be instructive to shew the method of calculating the parameter and the other quantities dependent on it, and for this purpose we must recur to the expression $VD = \frac{VC^2}{EV}$, an expression which, given in a specific form, becomes

parameter = *ordinate* × *ordinate* ÷ *abscissa* . . . . (E)

And when this equation is brought into the form of a rule, it is as follows:—

RULE.—*Multiply the semi-base or given ordinate by itself, and divide the product by the axis or abscissa, and the quotient will be the parameter sought.* Or more briefly thus:—*Divide the square of the ordinate by the abscissa for the parameter required.*

Example.—The axis and base of a parabola are each 30 inches; what is the parameter, and what is the distance of the focus and the directrix from the vertex of the curve?

Here the abscissa is 30 inches, and the ordinate or semi-base is 15 inches, consequently by the rule, it is

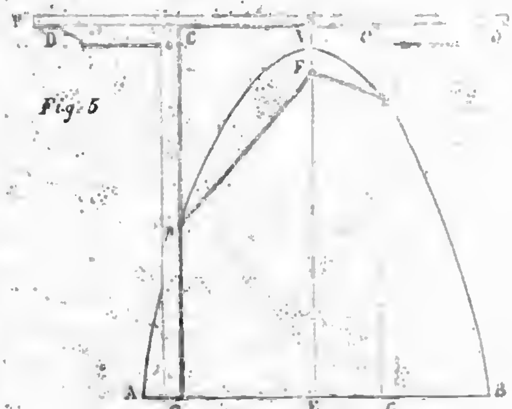
parameter = 15 × 15 ÷ 30 = 7·5 inches;

and the fourth part of this is 1·875 inches, the distance between the vertex and the focus, and also between the vertex and the points through which the directrix passes. It therefore follows, that if the axis be divided into ten equal parts of 3 inches each, the several radii by which the points in the curve are determined, are 4·875, 7·875, 10·875, 13·875, 16·875, 19·875, 22·875, 25·875, 28·875, and 31·875 inches respectively.

There are various other methods by which the parabolic curve may be described, but the best and most expeditious of all is that in which it is generated by continued motion, and one mode of generating it in this way is as follows:—

Let AB, fig. 5, be the base of the parabola and EV its axis; find the points F and n as in the last case, and through the point n so found draw the straight line PP, parallel to the base AB, and in consequence at right angles to the axis VE. Along the straight line PP let a ruler be fixed, and let one side of the square DCG, be brought into close contact with the lower side or edge of the ruler; then let a thread or cord FpG be taken equal in length to nE, the straight line which is made up of the axis VE and one-fourth of the parameter Vn, and let one end of this thread be fixed in the focus at F, while the other end is fixed at G, the lower extremity of the square DCG, the side CG being equal in length to the thread FpG. This being done, let CG, the side of the square, be made to coincide with nE, and let DC, the other side, be made to slide upon the ruler PP, while the thread is kept tight and close to the side CG, by means of the pin or pencil p; then the curve VpA, which is thus described, is one-half of the required parabola, and by only turning the square about, and causing it to slide in the opposite direction and on the other side of the fixed line nE, the curve VmB, which is traced

Fig. 5



out by the pencil at m, will be the other half of the parabola.

We have now to prove that the curve described by this motion is the common or conic parabola, and for this purpose we must suppose the square to slide along the rule PP until the points G and p become coincident in the point A, in which case the thread pF will be the hypothenuse of a right-angled triangle, of which the base is AE and perpendicular FE.* Now the length of the thread being by construction equal to nE=VE+Vn, the hypothenuse of the right-angled triangle just alluded to, must be equal to the same quantity: but VF is equal to Vn, consequently FE= VE—FV, and by the property of the right-angled triangle, we have,

$$(VE + Vn)^2 = AE^2 + (VE - FV)^2.$$

and from this by expanding and transposing the terms, we get

$$2\,VE \cdot Vn + 2\,VE \cdot FV = AE^2;$$

but FV=Vn by construction, therefore by substitution it is

$$4\,Vn \cdot VE = AE^2,$$

and since Vn is one-fourth of the parameter, the equation is

parameter × *abscissa* = *ordinate* × *ordinate*,

which, as we have already seen, is the equation of the common parabola.

There is another method of generating the curve by continued motion, which we think proper to introduce in this place, not that it is superior to the method just described, but because the same principle is applicable to the other sections also, thus reducing the system of conic construction to one uniform principle, which is beautifully calculated to shew the mutual relation of the several sections to one another.

Let AB (fig. 6), be the base of the parabola, and VE its axis, exactly as in the two cases preceding; upon the semi-base EB, and axis VE, describe the rectangular parallelogram

* We have omitted drawing a straight line from A to F in the figure, but the reader can easily supply it in his copy, which will render the steps of the reasoning more easily understood.